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Fighting Fish: enumerative properties

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Abstract. Fighting fish were very recently introduced by the authors as combinatorial structures made of square tiles that form two dimensional branching surfaces. A main feature of these fighting fish is that the area of uniform random fish of size n scales like $n^{5/4}$ as opposed to the typical $n^{3/2}$ area behavior of the staircase or direct convex polyominoes that they generalize.

In this extended abstract we concentrate on enumerative properties of fighting fish: in particular we provide a new decomposition and we show that the number of fighting fish with i left lower free edges and j right lower free edges is equal to

$$\frac{(2i+j-2)!(2j+i-2)!}{i!j!(2i-1)!(2j-1)!}.$$

These numbers are known to count rooted planar non-separable maps with $i+1$ vertices and $j+1$ faces, or two-stack-sortable permutations with respect to ascending and descending runs, or left ternary trees with respect to vertices with even and odd abscissa. However we have been unable until now to provide any explicit bijection between our fish and such structures. Instead we provide new refined generating series for left ternary trees to prove further equidistribution results.

Keywords: Enumerative combinatorics, exact formulas, bijections

1 Introduction

In a recent paper [7] we introduced a new family of combinatorial structures which we call *fighting fish* since they are inspired by the aquatic creatures known under the same name (see [the wikipedia page on Betta Splendens fish](#)). The easiest description of fighting fish is that they are built by gluing together unit squares of cloth along their edges in a directed way that generalize the iterative construction of directed convex polyominoes [8].

More precisely, we consider 45 degree tilted unit squares which we call *cells*, and we call the four edges of these cells *left upper edge*, *left lower edge*, *right upper edge* and

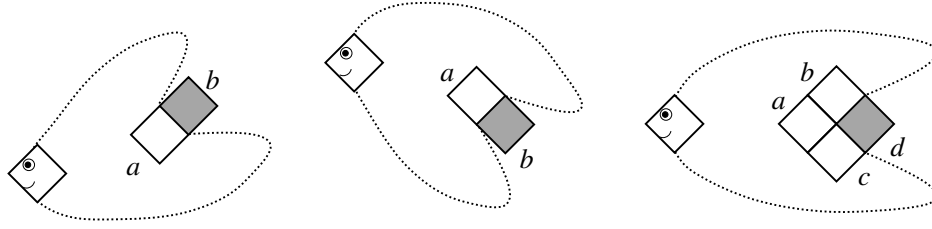


Figure 1: The three ways to grow a fish by adding a cell.

right lower edge respectively. We intend to glue cells along edges, and we call *free* an edge of a cell which it is not glued to an edge of another cell. All fighting fish are then obtained from an initial cell called the *head* by attaching cells one by one in one of the three following ways: (see Figure 1)

- Let a be a cell already in the fish whose right upper edge is free; then glue the left lower edge of a new cell b to the right upper edge of a .
- Let a be a cell already in the fish whose right lower edge is free; then glue the left upper edge of a new cell b to the right lower edge of a .
- Let a , b and c be three cells already in the fish and such that b (resp. c) has its left lower (resp. upper) edge glued to the right upper (resp. lower) side of a , and b (resp. c) has its right lower (resp. right upper) edge free; then simultaneously glue the left upper and lower edges of a new cell d respectively to the right lower edge of b and to the right upper edge of c .

While this description is iterative we are interested in the objects that are produced, independently of the order in which cells are added: a *fighting fish* is a collection of cells glued together edge by edge that *can* be obtained by the iterative process above. The *head* of the fighting fish is the only cell with two free left edges, its *nose* is the leftmost point of the head; a *final cell* is a cell with two free right edges, and the corresponding *tail* is its rightmost point; the *fin* is the path that starts from the nose of the fish, follows its border counterclockwise, and ends at the first tail it meets (see Figure 2(a)).

The *size* of a fighting fish is the number of lower free edges (which is easily seen to be equal to the number of upper free edges). Moreover, the *left size* (resp. *right size*) of a fighting fish is its number of left lower free edges (resp. right lower free edges). Clearly, the left and right size of a fish sum to its size. The *area* of a fighting fish is the number of its cells.

Examples of fighting fish are parallelogram polyominoes (aka staircase polyominoes), directed convex polyominoes, and more generally simply connected directed polyominoes in the sense of [8]. However, one should stress the fact that fighting fish are not

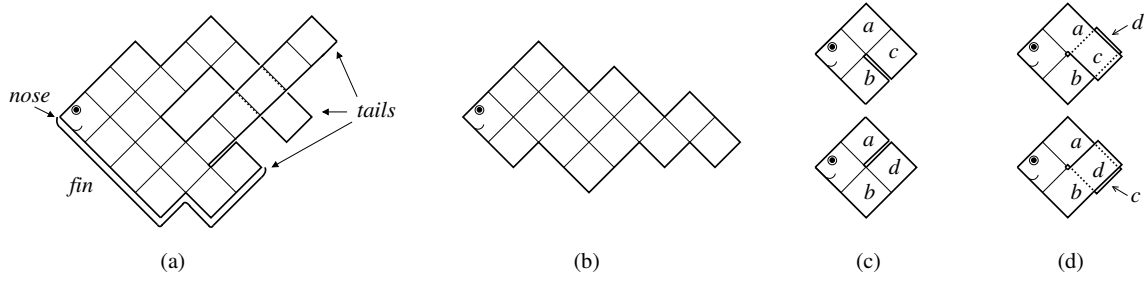


Figure 2: (a) A fighting fish which is not a polyomino; (b) A parallelogram polyomino; (c) The two fighting fish with area 4 that are not polyominoes; (d) Two different representations of the unique fighting fish with area 5 not fitting in the plane.

necessarily polyominoes because cells can be adjacent without being glued together and more generally cells are not constrained to fit in the plane and can cover each other, as illustrated by Figure 2(a) and 2(d). The two smallest fighting fish which are not polyominoes have size 5 and area 4: as illustrated by Figure 2(c), they are obtained by gluing a square a to the upper right edge of the head, a square b to the right lower edge of the head and either a square c to the right lower edge of a , or a square d to the upper right edge of b . The smallest fighting fish not fitting in the plane has size 6 and area 5, it is obtained by gluing both c to a and d to b : in the natural projection of this fighting fish onto the plane, squares c and d have the same image. Observe that we do not specify whether c is above or below d ; rather we consider that the surface has a branch point at vertex $c \cap d$ (see Figure 2(d)).

In [7] we obtained the generating series of fighting fish using essentially Temperley's approach, that is a decomposition in vertical slices. This allowed us to prove:

Theorem 1 ([7]). *The number of fighting fish with $n + 1$ lower free edges is*

$$\frac{2}{(n+1)(2n+1)} \binom{3n}{n} \quad (1.1)$$

We showed moreover that the average area of fighting fish of size n is of order $n^{5/4}$. This behavior suggests that, although fighting fish are natural generalizations of directed convex polyominoes, they belong to a different universality class: indeed the order of magnitude of the area of most classes of convex polyominoes is rather $n^{3/2}$ [11].

In the present extended abstract we explore further the remarkable enumerative properties of fighting fish. We propose in Section 2 a new decomposition that extends to fighting fish the classical wasp-waist decomposition of polyominoes [2]. Using the resulting equation we compute in Section 3.1 the generating series of fighting fish with respect to the numbers of left and right lower free edges, fin length and number of tails,

and use the resulting explicit parametrization to prove the following bivariate extension of Theorem 1:

Theorem 2. *The number of fighting fish with i left lower free edges and j right lower free edges is*

$$\frac{(2i+j-2)!(2j+i-2)!}{i!j!(2i-1)!(2j-1)!} = \frac{1}{ij} \binom{2i+j-2}{j-1} \binom{2j+i-2}{i-1}. \quad (1.2)$$

We also discuss in Section 3.2 several remarkable relations between fighting fish with marked points of various types. In particular we prove:

Theorem 3. *The number of fighting fish with i left lower free and j right lower free edges with a marked tail is*

$$\frac{(2i+2j-3)}{(2i-1)(2j-1)} \binom{2i+j-3}{j-1} \binom{2j+i-3}{i-1} \quad (1.3)$$

All these results confirm the apparently close relation of fighting fish to the well studied combinatorial structures known as *non separable planar maps* [4], *two stack sortable permutations* [12, 13, 1], and *left ternary trees* [5, 9]. The closest link appears to be between fighting fish and left ternary trees, that is, ternary trees whose vertices all have non negative abscissa in the natural embedding [10]. We prove in Section 4 the following theorem, which was conjectured in [7]:

Theorem 4. *The number of fighting fish with size n and fin length k is equal to the number of left ternary trees with n nodes, k of which are accessible from the root using only left and middle branches.*

We prove this theorem by an independent computation of the generating series of left ternary trees with respect to n and k (see Theorem 8), building on Di Francesco's method [6] for counting positively labeled trees. As discussed in Section 4 we conjecture that Theorem 4 extends to take into account the left and right size and the number of tails but we have only been able to prove this bijectively in the case of fighting fish with at most two tails, and in the case of fighting fish with h tails but at most $h+2$ lower edges that are not in the fin.

2 A wasp-waist decomposition

Theorem 5. *Let P be a fighting fish. Then exactly one of the following cases (A), (B1), (B2), (C1), (C2) or (C3) occurs:*

(A) P consists of a single cell;

(B) P is obtained from a smaller size fighting fish P_1 :

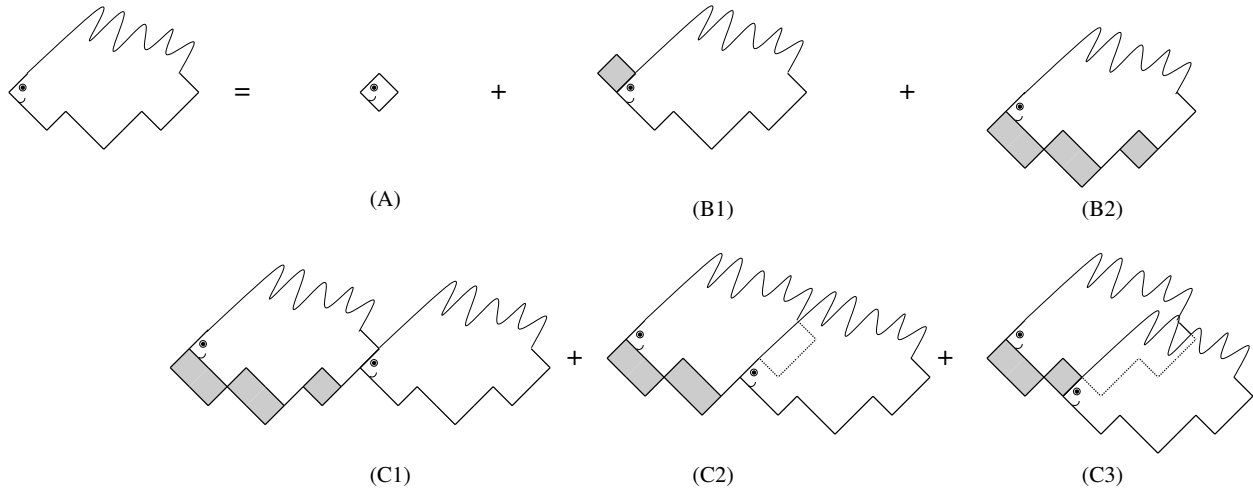


Figure 3: The wasp-waist decomposition.

(B1) by gluing the right lower edge of a new cell to the upper left edge of the head of P_1 (Figure 3 (B1));

(B2) by gluing every left edge of the fin of P_1 to the upper right edge of a new cell, and gluing the right lower edge and the upper left edge of all pairs of adjacent new cells (Figure 3 (B2));

(C) P is obtained from two smaller size fighting fish, P_1 and P_2 :

(C1) by performing to P_1 the operation described in (B2) and then gluing the upper left edge of the head of P_2 to the last edge of the fin of P_1 (Figure 3 (C1));

(C2) by choosing a right edge r on the fin of P_1 (last edge of the fin excluded) and gluing every left edge preceding r on the fin to the upper right edge of a new cell and, as above, gluing the right lower edge and the upper left edge of every pair of adjacent new cells; Then, gluing the upper left edge of the head of P_2 to r (Figure 3 (C2));

(C3) by choosing a left edge ℓ on the fin of P_1 and gluing every left edge of the fish fin preceding ℓ (included) to the upper right edge of a new cell and, as above, gluing the right lower edge and the upper left edge of every pair of adjacent new cells; Then, gluing the upper left edge of the head of P_2 to the right lower edge of the cell glued to ℓ (Figure 3 (C3)).

Moreover each of the previous operations, when applied to arbitrary fighting fish P_1 and if necessary P_2 , produces valid a fighting fish.

Observe that Cases (A), (B1) and (B2) could have been alternatively considered as degenerate cases of Case (C1) where P_1 or P_2 would be allowed to be empty. Staircase

polyominoes are exactly the fighting fish obtained using only Cases (A), (B1), (B2) and (C1).

Proof. Omitted (Appendix A). □

Let $P(t, y, a, b; u) = \sum_P t^{\text{size}(P)-1} y^{\text{tails}(P)-1} a^{\text{rsize}(P)-1} b^{\text{lsize}(P)-1} u^{\text{fin}(P)-1}$ denote the generating series of fighting fish with variables t, y, a, b and u respectively marking the size, the number of tails, the right size, the left size, the fin length, all decreased by one.

Corollary 1. *The generating series $P(u) \equiv P(t, y, a, b; u)$ of fighting fish satisfies the equation*

$$P(u) = tu(1 + aP(u))(1 + bP(u)) + ytabuP(u) \frac{P(1) - P(u)}{1 - u}. \quad (2.1)$$

Proof. This is a direct consequence of the previous theorem, details are omitted (Appendix A). □

3 Enumerative results for fish

3.1 The algebraic solution of the functional equation

The equation satisfied by fighting fish is a combinatorially funded polynomial equation with one catalytic variable: this class of equations was thoroughly studied by Bousquet-Mélou and Jehanne [3] who proved that they have algebraic solutions.

Theorem 6. *Let $B \equiv B(t; y, a, b)$ denote the unique power series solution of the equation*

$$B = t \left(1 + y \frac{abB^2}{1 - abB^2} \right)^2 (1 + aB)(1 + bB). \quad (3.1)$$

Then the generating series $P(1) \equiv P(t; y, a, b, 1)$ of fighting fish can be expressed as

$$P(1) = B - \frac{yabB^3(1 + aB)(1 + bB)}{(1 - abB^2)^2}. \quad (3.2)$$

This theorem easily implies Theorem 2 using Lagrange inversion (Appendix C).

Proof of Theorem 6. Our proof follows closely the approach of [3], so we omit the details (Appendix B) and only present the strategy: Rewrite Equation (2.1) as

$$(u - 1)P(u) = tu(u - 1)(1 + aP(u))(1 + bP(u)) + ytuabP(u)(P(u) - P(1)). \quad (3.3)$$

and take the derivative with respect to u :

$$\begin{aligned} P(u) - t(2u - 1)(1 + aP(u))(1 + bP(u)) - ytabP(u)(P(u) - P(1)) \\ = -\frac{\partial}{\partial u} P(u) \cdot (u - 1 - tu(u - 1)(a + b + 2abP(u)) - ytuab(2P(u) - P(1))) \end{aligned}$$

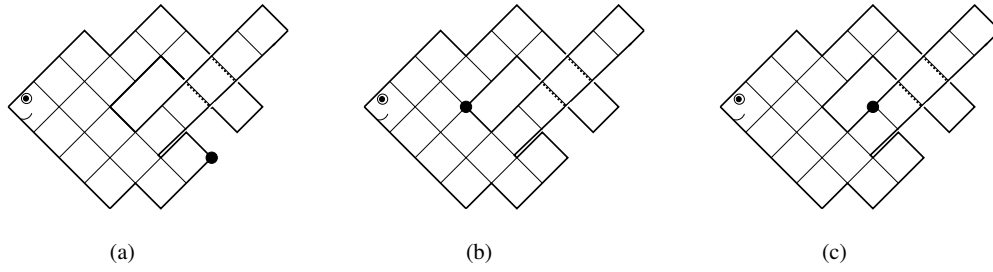


Figure 4: Fish with marked points: (a) a tail, (b) a branch point, (c) an upper flat point.

Now there clearly exists a unique power series U that cancels the second factor in the right hand side of the previous equation: U is the unique power series root of the equation

$$U - 1 = tU(U - 1)(a + b + 2abP(U)) + ytabU(2P(U) - P(1)). \quad (3.4)$$

Since U must also cancel the left hand side,

$$P(U) = t(2U - 1)(1 + aP(U))(1 + bP(U)) + ytabP(U)(P(U) - P(1)), \quad (3.5)$$

and, for $u = U$, Equation (3.3) reads

$$(U - 1)P(U) = tU(U - 1)(1 + aP(U))(1 + bP(U)) + ytUabP(U)(P(U) - P(1)). \quad (3.6)$$

Solving the resulting system of 3 equations for the three unknown U , $P(U)$ and $P(1)$ yields the theorem, with $P(U) = B$. \square

The full series $P(u)$ is clearly algebraic of degree at most 2 over $\mathbb{Q}(u, B)$, but it admits in fact a parametrization directly extending the one of the theorem.

Corollary 2. *Let $B(u)$ be the unique power series solution of the equation:*

$$B(u) = tu \left(1 + aB(u) + yaB(u) \frac{bB(1 + aB)}{1 - abB^2} \right) \left(1 + bB(u) + ybB(u) \frac{aB(1 + bB)}{1 - abB^2} \right) \quad (3.7)$$

then

$$P(u) = B(u) - yabB(u)^2 B \frac{(1 + aB)(1 + bB)(1 - abB^2 + yabB^2)}{(1 - abB^2)^2(1 - abB(u)B + yabB(u)B)}.$$

3.2 Fighting fish with marked points

Let $P^<$ denote the generating series of fish with a marked branched point, $P^>$ the generating series of fish with a marked tail, $2P^-$ the generating series of fish with a marked

flat point, that is, a marked point which is neither the head nor a tail nor a branched point (observe that each fish has the same number of upper and lower flat points, hence the factor 2). The generating series of fighting fish with a marked point is then:

$$P(1) + 2P^- + P^> + P^< = 2\frac{t\partial}{\partial t}(tP(1)). \quad (3.8)$$

From the fact that there is always one more tail than branch point we have

$$P(1) + P^< = P^> \quad (3.9)$$

so that we also have

$$P^- + P^> = P(1) + P^< + P^- = \frac{t\partial}{\partial t}(tP(1)). \quad (3.10)$$

Fighting fish with a marked tail can also be counted thanks to the variable y :

$$P^< = \frac{y\partial}{\partial y}P(1), \quad \text{or} \quad P^> = \frac{y\partial}{\partial y}(yP(1)). \quad (3.11)$$

Observe that derivating Equation (3.3) with respect to y instead of u yields the same coefficient for the derivative of $P(u)$, which cancels for $u = U$. This simplification leads to the remarkable relations:

$$P^< = y\frac{\partial}{\partial y}P(1) = P(U) - P(1), \quad \text{and} \quad P^> = P(U). \quad (3.12)$$

This relation allows to use bivariate Lagrange inversion on the parametrization $P(U) = B$ in Theorem 6 to prove Theorem 3, we omit the details (Appendix C).

Similarly derivating Equation (3.3) with respect to t and taking $u = U$ yields:

$$U = \frac{1}{1-V} \quad \text{where} \quad V = ytab\frac{t\partial}{\partial t}P(1) = ytab(P^- + P^<). \quad (3.13)$$

Equations (3.12) and (3.13) admit direct combinatorial interpretations (Appendix D).

4 Fighting fish and left ternary trees: the fin/core relation

A *ternary tree* is a finite tree which is either empty or contains a root and three disjoint ternary trees called the left, middle and right subtrees of the root. Given a initial root label j , a ternary tree can be naturally embedded in the plane in a deterministic way: the root has abscissa j and the left (resp. middle, right) child of a node with abscissa $i \in \mathbb{Z}$ has abscissa $i - 1$ (resp. i , $i + 1$). A *j -positive tree* is a ternary tree whose nodes all have non positive abscissa; 0-positive trees were first introduced in the literature with the

name *left ternary tree* [5, 9] (in order to be coherent with these works one should orient the abscissa axis toward the left).

It is known that the number of left ternary trees with i nodes at even position and j nodes at odd position is given by Formula (1.2) [5, 9]. In order to refine this result we introduce the following new parameters on left ternary trees:

- Let the *core* of a ternary tree T be the largest subtree including the root of T and consisting only of left and middle edges.
- Let a *right branch* of a ternary tree be a maximal sequence of right edges.

In order to prove Theorem 4 we compute the generating series of j -positive trees according to the number of nodes and nodes in the core.

4.1 A refined enumeration of j -positive trees

In this section we implicitly take $y = a = b = 1$ in all generating series.

Let T , B and X be the unique formal power series solutions of

$$T = 1 + tT^3, \quad \text{and} \quad B = tT^2 \quad \text{and} \quad X = B(1 + X + X^2).$$

Observe that B coincide with the series $B(t; 1, 1, 1, 1)$ of the previous sections and that,

$$T = \frac{1}{1 - B} = \frac{1 + X + X^2}{1 + X^2}, \quad \text{and} \quad T - 1 = BT = \frac{X}{1 + X^2}, \quad \text{and} \quad B = \frac{X}{1 + X + X^2}.$$

Building on Di Francesco's educated guess and check approach [6], Kuba obtained a formula for j -positive trees reads:

Theorem 7 ([10, 6]). *The generating series $T_j = T_j(t; 1, 1, 1, 1)$ is given for all $j \geq 0$ by the explicit expression:*

$$T_j = T \frac{(1 - X^{j+5})(1 - X^{j+2})}{(1 - X^{j+4})(1 - X^{j+3})}.$$

Define moreover

$$T(u) = 1 + tuT(u)^2T \quad \text{and} \quad B(u) = tuT(u)^2$$

so that

$$T(1) = T, \quad \text{and} \quad T(u) = 1 + B(u)T, \quad \text{and} \quad B(u) = (T(u) - 1)(1 - B)$$

Observe that this $B(u)$ coincide with the series $B(t; 1, 1, 1, u)$ of the previous sections, so that the generating series of fighting fish according to the size and the fin length, given by Corollary 2, can be written as (recall that here $y = a = b = 1$)

$$1 + P(u) = 1 + B(u) - B(u)^2 \frac{B}{(1 - B)^2} = T(u)(1 + B) - T(u)^2 B. \quad (4.1)$$

Theorem 8. The generating series $T_j(u) \equiv T_j(t; 1, 1, 1, u)$ is given for $j \geq -1$ by

$$T_j(u) = T(u) \frac{H_j(u)}{H_{j-1}(u)} \frac{1 - X^{j+2}}{1 - X^{j+3}}$$

where for all $j \geq -2$,

$$H_j(u) = (1 - X^{j+1})XT(u) - (1 + X)(1 - X^{j+2}).$$

Corollary 3. The number of left ternary trees with n vertices, k of which belong to the core, is equal to the number of fighting fish of size n with fin length k .

Proof of Corollary 3. This is a simple computation:

$$T_0(u) = T(u) \frac{H_0(u)}{H_{-1}(u)} \frac{1 - X^2}{1 - X^3}.$$

By definition $H_{-1} = -(1 + X)(1 - X)$ and $H_{-2}(u) = (X - 1)T(u)$, so that

$$T_0(u) = T(u) \frac{(1 - X)XT(u) - (1 + X)(1 - X^2)}{-(1 - X^2)} \frac{1 - X^2}{1 - X^3} = -T(u)^2B + T(u)(1 + B).$$

which coincide with Equation (4.1). □

Proof of Theorem 8. In order to prove the theorem it is sufficient to show that the series given by the explicit expression satisfies for all $j \geq -1$ the equation:

$$T_j(u) = 1 + tuT_{j+1}(u)T_j(u)T_{j-1} \quad (4.2)$$

where T_j is given by Theorem 7, with the convention that $T_{-2} = 0$: indeed the system of Equations (4.2) clearly admits the generating series of j -positive ternary trees as unique power series solutions. The case $j = -1$ is immediate:

$$T_{-1}(u) = T(u) \frac{H_{-1}(u)}{H_{-2}(u)} \frac{1 - X}{1 - X^2} = 1.$$

Let now $j \geq 0$, then the right hand side of Equation (4.2) reads

$$\begin{aligned} & 1 + tuT(u)^2T \cdot \frac{H_{j+1}(u)}{H_{j-1}(u)} \cdot \frac{1 - X^{j+4}}{1 - X^{j+3}} \cdot \frac{1 - X^{j+1}}{1 - X^{j+4}} \\ &= \frac{H_{j-1}(u)(1 - X^{j+3}) + (T(u) - 1)H_{j+1}(u)(1 - X^{j+1})}{H_{j-1}(u)(1 - X^{j+3})} \end{aligned}$$

and we want to show that this is equal to

$$\frac{T(u)H_j(u)(1 - X^{j+2})}{H_{j-1}(u)(1 - X^{j+3})}.$$

Now

$$\begin{aligned} H_{j-1}(u)(1 - X^{j+3}) &= (1 - X^{j+3})(1 - X^j)XT(u) - (1 + X)(1 - X^{j+1})(1 - X^{j+3}), \\ -H_{j+1}(u)(1 - X^{j+1}) &= -(1 - X^{j+1})(1 - X^{j+2})XT(u) + (1 + X)(1 - X^{j+3})(1 - X^{j+1}), \\ T(u)H_{j+1}(u)(1 - X^{j+1}) &= (1 - X^{j+1})(1 - X^{j+2})XT(u)^2 - (1 + X)(1 - X^{j+3})(1 - X^{j+1})T(u), \end{aligned}$$

while

$$T(u)H_j(u)(1 - X^{j+2}) = (1 - X^{j+2})(1 - X^{j+1})XT(u)^2 - (1 + X)(1 - X^{j+2})^2T(u).$$

The coefficients of $T(u)^2$ and $T(u)^0$ are clearly matching. Upon expanding all contributions to the coefficient of $T(u)$ in power of X , the various terms are directly seen to match as well. \square

4.2 A refined conjecture

In view of Theorem 2 and Theorem 4, it is natural to look for a common generalization. Indeed one can even take the number of tails into account:

Conjecture 1. *The number of fighting fish with size n , fin length k , having h tails, with i left lower free edges and j right lower free edges is equal to the number of left ternary trees with n nodes, core size k , having h right branches, with $i + 1$ non root nodes with even abscissa and j nodes with odd abscissa.*

This conjecture naturally calls for a bijective proof, however we have been unable to provide such a proof, except in two specific cases:

- The case of left ternary trees with at most one right branch, which are in bijections with fighting fish with at most two tails for all values of n , k , i and j .
- The case of left ternary trees with h right branches and at most $h + 2$ vertices, which are in bijection with fighting fish with h tails and $h + 2$ lower edges that are not in the fin.

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Figure 5: A real fighting fish.

A Proof of Theorem 5 and Corollary 1

Proof of the theorem. The operations described in Theorem 5 produce valid fighting fish: indeed given incremental growths of P_1 and P_2 we obtain a valid incremental growth of P upon starting from the new head, growing the head of P_1 interleaving the next steps of the growth of P_1 with insertions of the new cells: each new cell is to be inserted just before the fin cell it will be attached to; when this is done, the head of the fish P_2 can be attached on P_1 and the rest of P_2 growth from there.

It thus remains to show that every fighting fish of size greater than 2 can be uniquely obtained by applying one of the operations (B) or (C) to fish of smaller size.

In order to prove the result let us describe how to decompose a fish P which is not reduced to a cell. In order to do this we need two further definitions: First let us call *cut edge* of P any common side e of two cells of P such that cutting P along e yields two connected components. Second let the set of *fin cells* of P be the set of cells incident to a left edge of the fin: the head of P is always a fin cell and the other fin cells have non-free left upper sides (since their left lower sides are free and they must be attached by a left side).

Now the decomposition is as follows:

- First mark the head of P as *removable* and consider the other fin cells iteratively from left to right: mark them as *removable* as long as their left upper side is not a cut edge of P . Let $R(P)$ be the set of removable cells of P .
- If all fin cells are marked as removable then removing these cells yields a fighting fish $P_1 = P \setminus R(P)$, and applying the construction of Case (B2) to P_1 gives P back. Conversely any fish produced as in Case (B2) has all its fin cells removable.
- Otherwise let c be the first fin cell which is not removable. Upon cutting the

left upper side e of c , two components are obtained: let P_2 be the component containing c and let \bar{P}_1 be the other component, which contains by construction all the removable cells of P . Using the incremental construction of fighting fish one easily check that $P_1 = \bar{P}_1 \setminus R(P)$ is a (possibly empty) fighting fish, and P_2 is a non-empty fighting fish.

- If P_1 is empty then applying the construction of Case (B2) to P_1 yields P back, and conversely all fish produced as in Case (B2) have a decomposition with P_1 empty.
- Otherwise the edge e corresponds to a right lower side \bar{e}_1 on the fin of \bar{P}_1 , or equivalently to an edge e_1 of the fin of P_1 : if \bar{e}_1 is a side of a removable cell of P then e_1 is the right upper edge of this cell, which is a right lower edge on the fin of P_1 (this corresponds to Case (C3)); otherwise $e_1 = e$ is a right lower edge on the fin of P_1 (this corresponds to Case (C1) or (C2) depending whether e_1 is the rightmost edge on the fin of P_1 or not).

□

Proof of the corollary. The wasp-waist decomposition of Theorem 5 is readily translated into the following functional equation:

$$\begin{aligned}
 P(u) &= tu + tubP(u) + tuaP(u) + tuabP(u)^2 + \\
 &\quad ytabP(u) \sum_{P_1} t^{\text{size}(P_1)-1} y^{\text{tails}(P_1)-1} a^{\text{rsize}(P_1)-1} b^{\text{lsz}(P_1)-1} \left(u + \dots + u^{\text{fin}(P_1)-1} \right) \\
 &= tu(1 + aP(u))(1 + bP(u)) + ytabuP(u) \frac{P(1) - P(u)}{1 - u},
 \end{aligned}$$

where the only difficult point is to observe that given a pair (P_1, P_2) of non empty fighting fish with $\text{fin}(P_1) = k + 1$, Cases (C2) and (C3) together produce k fighting fish with fin size $2, 3, \dots, k + 1$ respectively. □

B Proof of Theorem 6

Let us resume the proof from the system formed by Equations (3.6), (3.4) and (3.5).

Comparing Equation (3.6) and Equation (3.5) multiplied by U we immediately deduce the simpler relation

$$P(U) = tU^2(1 + aP(U))(1 + bP(U)). \quad (\text{B.1})$$

Now comparing Equation (3.6) and Equation (3.4) multiplied by $P(U)$ yields, up to canceling a factor tU ,

$$(U - 1)(1 + (a + b)P(U) + abP(U)^2) = (U - 1)P(U)(a + b + 2abP(U)) + yabP(U)^2,$$

that is

$$U = 1 + y \frac{abP(U)^2}{1 - abP(U)^2} \quad (\text{B.2})$$

In view of Equations (B.1) and (B.2), $P(U)$ is the unique formal power series solution of the equation:

$$P(U) = t \left(1 + y \frac{abP(U)^2}{1 - abP(U)^2} \right)^2 (1 + aP(U))(1 + bP(U)). \quad (\text{B.3})$$

In other terms $P(U) = B$ as defined in Theorem 6. Now using Equation (B.2) to eliminate U in Equation (3.6), and canceling a factor $yabP(U)$ we have:

$$\frac{P(U)^2}{1 - abP(U)^2} = tU \frac{P(U)}{1 - abP(U)^2} (1 + aP(U))(1 + bP(U)) + tU(P(U) - P(1)).$$

Using Equation (B.1) to expand a factor $P(U)$ in the left hand side, and canceling a factor tU , this equation can be rewritten as:

$$\frac{P(U)U(1 + aP(U))(1 + bP(U))}{1 - abP(U)^2} = \frac{P(U)}{1 - abP(U)^2} (1 + aP(U))(1 + bP(U)) + (P(U) - P(1)). \quad (\text{B.4})$$

In other words:

$$P(U) - P(1) = (U - 1) \frac{P(U)(1 + aP(U))(1 + bP(U))}{1 - abP(U)^2}$$

and using again Equation (B.1),

$$P(U) - P(1) = \frac{yabP(U)^3(1 + aP(U))(1 + bP(U))}{(1 - abP(U)^2)^2}. \quad (\text{B.5})$$

Finally

$$P(1) = P(U) - \frac{yabP(U)^3(1 + aP(U))(1 + bP(U))}{(1 - abP(U)^2)^2}, \quad (\text{B.6})$$

which concludes the proof of the theorem using $P(U) = B$.

C Proof of the bivariate formulas

In this proof we implicitly set $y = 1$ in all series. Theorems 2 and 3 can be derived by bivariate Lagrange inversion on the expression of $P(1)$ in terms of the series

$$\bar{R} = \frac{aB(1 + bB)}{1 - abB^2} \quad \text{and} \quad \bar{S} = \frac{bB(1 + aB)}{1 - abB^2}.$$

Indeed

$$B = t \frac{(1 + aB)(1 + bB)}{(1 - abS^2)^2} = t(1 + \bar{R})(1 + \bar{S}), \quad (\text{C.1})$$

so that \bar{R} and \bar{S} satisfy

$$\begin{cases} \bar{R} &= ta(1 + \bar{R})(1 + \bar{S})^2 \\ \bar{S} &= tb(1 + \bar{R})^2(1 + \bar{S}). \end{cases} \quad (\text{C.2})$$

Indeed Equation (3.2) then rewrites as

$$P(1) = B - abB^3 \frac{(1 + aB)(1 + bB)}{(1 - abB^2)^2} = t(1 + \bar{R})(1 + \bar{S})(1 - \bar{R}\bar{S}). \quad (\text{C.3})$$

Given a system $\{A_1 = a_1\Phi_1(A_1, A_2), A_2 = a_2\Phi_2(A_1, A_2)\}$ the bivariate Lagrange inversion theorem states that for any function $F(x_1, x_2)$,

$$\begin{aligned} [a_1^{n_1} a_2^{n_2}]F(A_1, A_2) &= \frac{1}{n_1 n_2} [x_1^{n_1-1} x_2^{n_2-1}] \left(\frac{\partial^2 F(x_1, x_2)}{\partial x_1 \partial x_2} \Phi_1(x_1, x_2)^{n_1} \Phi_2(x_1, x_2)^{n_2} \right. \\ &\quad + \frac{\partial F(x_1, x_2)}{\partial x_1} \frac{\partial \Phi_1(x_1, x_2)^{n_1}}{\partial x_2} \Phi_2(x_1, x_2)^{n_2} \\ &\quad \left. + \frac{\partial F(x_1, x_2)}{\partial x_2} \frac{\partial \Phi_2(x_1, x_2)^{n_2}}{\partial x_1} \Phi_1(x_1, x_2)^{n_1} \right) \end{aligned}$$

In other words,

$$[a_1^{n_1} a_2^{n_2}]F(A_1, A_2) = \frac{1}{n_1 n_2} [x_1^{n_1-1} x_2^{n_2-1}] \Phi_1(x_1, x_2)^{n_1} \Phi_2(x_1, x_2)^{n_2} H, \text{ where}$$

$$H = \frac{\partial^2 F(x_1, x_2)}{\partial x_1 \partial x_2} + n_1 \frac{\partial F(x_1, x_2)}{\partial x_1} \frac{\partial \Phi_1(x_1, x_2)}{\partial x_2} \frac{1}{\Phi_1(x_1, x_2)} + n_2 \frac{\partial F(x_1, x_2)}{\partial x_2} \frac{\partial \Phi_2(x_1, x_2)}{\partial x_1} \frac{1}{\Phi_2(x_1, x_2)}.$$

Setting $t = 1$ and applying the bivariate Lagrange inversion formula to the function $B(\bar{R}, \bar{S})$ in Equation (C.1), where $\bar{R} = a\Phi_1(\bar{R}, \bar{S})$ and $\bar{S} = b\Phi_2(\bar{R}, \bar{S})$ as defined in system (C.2), yields

$$\begin{aligned} [a^{i-1} b^{j-1}]B &= \frac{1}{(i-1)(j-1)} [\bar{R}^{i-2} \bar{S}^{j-2}] \left((1 + \bar{R})^{i+2j-3} (1 + \bar{S})^{2i+j-3} (2i + 2j - 3) \right) \\ &= \frac{(2i + 2j - 3)}{(i-1)(j-1)} \binom{2j+i-3}{i-2} \binom{2i+j-3}{j-2} \\ &= \frac{(2i + 2j - 3)}{(2i-1)(2j-1)} \binom{2j+i-3}{i-1} \binom{2i+j-3}{j-1}. \end{aligned}$$

This proves Theorem 3. Now, apply the bivariate Lagrange inversion formula to the function $P(1)$ in Equation (C.3): it holds

$$[a^{i-1}b^{j-1}]P(1) = \frac{1}{(i-1)(j-1)}[\bar{R}^{i-2}\bar{S}^{j-2}] \left((1 + \bar{R})^{i+2j-3}(1 + \bar{S})^{2i+j-3} (-4 + 4\bar{R}\bar{S} + 2i(1 - 2\bar{R}\bar{S} - \bar{S}) + 2j(1 - 2\bar{R}\bar{S} - \bar{R})) \right).$$

By extracting coefficients of $\bar{R}^{i-2}\bar{S}^{j-2}$ yields

$$[a^{i-1}b^{j-1}]P(1) = \frac{1}{(i-1)(j-1)} \left(2(i+j-2) \binom{2j+i-3}{i-2} \binom{2i+j-3}{j-2} - 4(i+j-1) \binom{2j+i-3}{i-3} \binom{2i+j-3}{j-3} - 2j \binom{2j+i-3}{i-3} \binom{2i+j-3}{j-2} - 2i \binom{2j+i-3}{i-2} \binom{2j+i-3}{j-3} \right).$$

Manipulating and summing all the binomial coefficients it results

$$[a^{i-1}b^{j-1}]P(1) = \frac{1}{ij} \binom{2j+i-2}{i-1} \binom{2i+j-2}{j-1}.$$

D Bijective interpretations

D.1 A bijective proof of the relation $P^> = P(U)$

Proposition 1. *There is a bijection between*

- *fighting fish with a marked tail having $i + 1$ left lower free edges and $j + 1$ right lower free edges,*
- *and pairs (P, S) where P is a fighting fish with fin size $k + 1$ and S is a k -uple (U_1, \dots, U_k) of sequences $U_i = (V_{i,1}, \dots, V_{i,j_i})$ of fish that are marked on an upper flat point or a branch point, such that the total number of left lower free edges and right lower free edges are respectively $i + 1$ and $j + 1$.*

The bijection is illustrated by Figure 6.

Sketch of proof. Given a pair (P, S) as above, mark the first tail of P , then slice P above each inner point of its fin, cut each $V_{i,j}$ at its nose and marked point and inflate it vertically to match the width of P above the i th point x_i of its fin, and insert the inflated sequence U_i between the slice before and after x_i . This produce a fighting fish P' with a

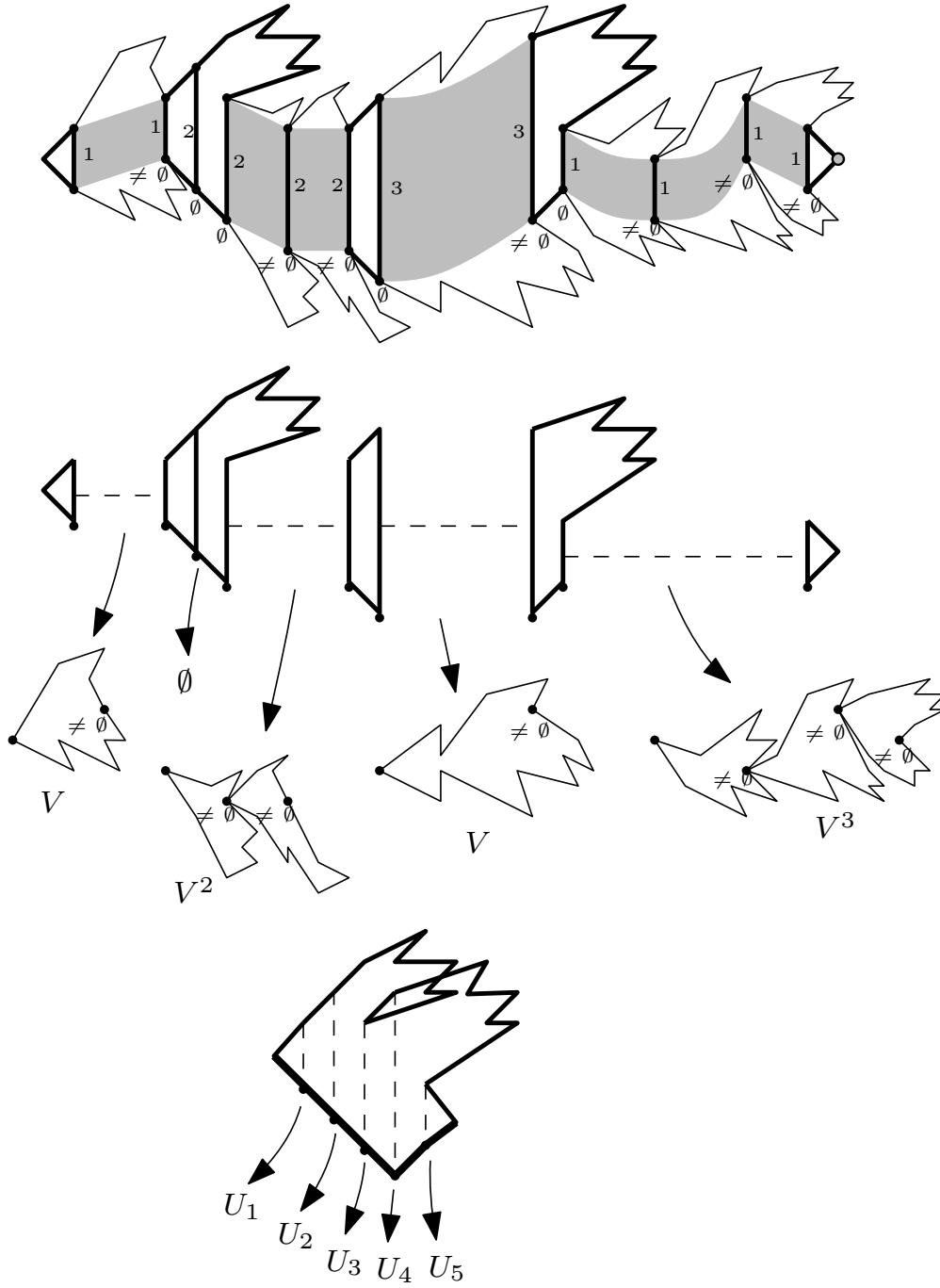


Figure 6: The bijective interpretation of $P^> = P(U)$.

marked tail. The fact that the marked point of each $V_{i,j}$ is an upper flat point or a branch point ensures that the resulting point in P' is a branch point.

Conversely a fish P' with a marked tail can be decomposed upon traveling along the *spine* connecting the tail to the nose and cutting the fish in slices: Starting from the tail,

1. travel to the left along the current lower free edge to reach a point x
2. if x is a flat point then a new slice of P is obtained above the edge that has been traversed; resume at step 1;
3. otherwise x is a branch point, then let ℓ denote the length of the shortest vertical cut above x that separates the nose and the tail:
 - (a) travel along the spine until the length of the vertical cut returns to the value ℓ for the first time, above a new lower point x : the resulting slice gives the next factor $V_{i,j}$ in the decomposition;
 - (b) if the new point x is again a branch point then resume at the previous step,
 - (c) otherwise x is a lower flat point and a new factor U_i has been completed; resume at step 1.

The proof that the two constructions above are inverse one of the other is omitted. \square

D.2 A bijective proof of the relation $V = ytab(\Delta P)(U)$

Let $(\Delta P)(u) = u \frac{P(u) - P(1)}{u - 1} = P(u^k \rightarrow u + \dots + u^k)$. Then $(\Delta P)(u)$ is the generating series of fighting fish with a marked edge on the fin, where u marks the distance between the nose and the endpoint of the marked edge.

Proposition 2. *There is a bijection between*

- *fighting fish with a marked branch or flat lower point having $i + 1$ left lower free edges and $j + 1$ right lower free edges,*
- *and pairs (P, S) where P is a fighting fish with a marked edge on the fin at distance k from the nose and S is a k -uple (U_1, \dots, U_k) of sequences $U_i = (V_{i,1}, \dots, V_{i,j_i})$ of fish that are marked on an upper flat point or a branch point, such that the total number of left lower free edges and right lower free edges are respectively $i + 1$ and $j + 1$.*

Sketch of proof. This bijection is based on the same decomposition as the previous one. The only difference is that the substitution of U factors is not made along the whole fin of the fish: as a result the marked point is a flat lower point or a branch point instead of being a tail. \square

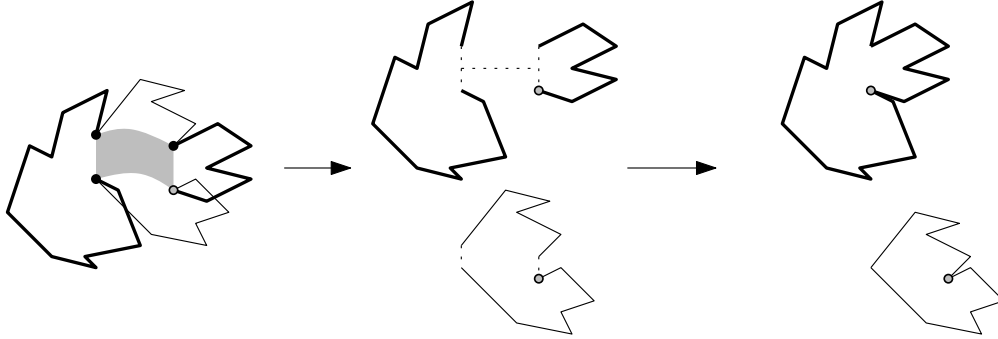


Figure 7: The bijective interpretation of $ytab P^< = V^2 = (ytab(P^- + P^<))^2$.

D.3 A bijective proof of the relation $ytab P^< = V^2$

Proposition 3. *There is a bijection between*

- *fighting fish a marked branch point having $i + 1$ left lower free edges and $j + 1$ right lower free edges,*
- *and pairs of fighting fish (P_1, P_2) marked on a branch or lower flat point, such that the total number of left lower free edges and right lower free edges are respectively $i + 1$ and $j + 1$.*

The bijection is illustrated by Figure 7.

Proof. Let P be a fighting fish with a marked branch point x , and let ℓ denote the length of the shortest vertical cut segment above x . Then the decomposition is obtained by cutting P at this vertical cut segment and at the first vertical cut segment of the same length that is found along the spine from x toward the nose of P . \square

D.4 A bijective proof of the relation $P^> = tU^2(1 + aP^>)(1 + bP^>)$

This relation follows from the wasp-waist decomposition of fighting fish upon substituting $u = U$: the relation $P^> = P(U)$ and $ytab U(\Delta P)(U) = V$ indeed immediately leads to $P^> = tU(1 + aP^>)(1 + bP^>) + VP^>$, from which one concludes by iteration.